MATH5360 Game Theory
Exercise 2

Assignment 2: 1(b), 2(a)(c)(f) (Due: 16 March 2020 (Monday))

1. Solve the following primal problems. Then write down the dual problems and the solutions to the dual problems.

   (a)
$$\begin{array}{rl} \max & f = 3y_1 + 5y_2 + 4y_3 + 12 \\ \text{subject to} & 3y_1 + 2y_2 + 2y_3 \le 15 \\ & 4y_2 + 5y_3 \le 24 \end{array}$$

   (b)
$$\begin{array}{rl} \max & f = 2y_1 + 4y_2 + 3y_3 + y_4 \\ \text{subject to} & 3y_1 + y_2 + y_3 + 4y_4 \le 12 \\ & y_1 - 3y_2 + 2y_3 + 3y_4 \le 7 \\ & 2y_1 + y_2 + 3y_3 - y_4 \le 10 \end{array}$$

2. Solve the zero sum games with the following game matrices, that is find the value of the game, a maximin strategy for the row player and a minimax strategy for the column player.

   (a) $\begin{pmatrix} 2 & -3 & 3 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}$

   (b) $\begin{pmatrix} 3 & 1 & -5 \\ -1 & -2 & 6 \\ -2 & -1 & 3 \end{pmatrix}$

   (c) $\begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix}$

   (d) $\begin{pmatrix} 2 & 0 & -2 \\ -1 & -3 & 3 \\ -2 & 2 & 0 \end{pmatrix}$

   (e) $\begin{pmatrix} 1 & -1 & 1 \\ -2 & 0 & -1 \\ 1 & -2 & 2 \\ -1 & 1 & -2 \end{pmatrix}$

   (f) $\begin{pmatrix} -3 & 2 & 0 \\ 1 & -2 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & -3 \end{pmatrix}$

3. Prove that if C_1 and C_2 are convex sets in $\mathbb{R}^n$, then the following sets are also convex.

   (a) $C_1 \cap C_2$

   (b) $C_1 + C_2 = \{\mathbf{x}_1 + \mathbf{x}_2 : \mathbf{x}_1 \in C_1, \mathbf{x}_2 \in C_2\}$

4. Let A be an $m \times n$ matrix. Prove that the set of maximin strategies for the row player of A is convex.

5. Let C be a convex set in $\mathbb{R}^n$ and $\mathbf{x}, \mathbf{y} \in C$. Let $\mathbf{z} \in \mathbb{R}^n$ be a point on the straight line joining $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{z}$ is orthogonal to $\mathbf{x} - \mathbf{y}$.

   (a) Find $\mathbf{z}$ in terms of $\mathbf{x}$ and $\mathbf{y}$.

(b) Suppose $\langle \mathbf{x}, \mathbf{y} \rangle < 0$. Prove that $\mathbf{z} \in C$.

6. Let A be an $m \times n$ matrix with column vectors $\mathbf{a}_1^T, \mathbf{a}_2^T, \cdots, \mathbf{a}_n^T$. Let $\nu_c(A)$ be the column value of A and let

$$C = \text{Conv}(\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\})$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ is the standard basis for $\mathbb{R}^m$. Prove that if $\nu_c(A) \leq 0$, then $\mathbf{0} \in C$.